Weakly Nehari Functions, Hyperbolic Convexity and John Disks

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Abstract

In this paper, we extend some recent results by Hag and Hag regarding a generalized *Nehari* class. In particular, we give a description of this new class in terms of the hyperbolic metric, and characterize the unbounded functions in an analogue of a theorem of Gehring and Pommerenke. We also derive several sharp distortion theorems and estimates on the pre-Schwarzian that are used to study John disks.

1. INTRODUCTION

Among univalent functions f defined in the unit disk $\mathbb{D}$, the so called Nehari class N, given by the condition

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}\,, \tag{1}$$

has been studied in fairly deep detail since Nehari's original result [N] that related the size of the Schwarzian derivative $Sf = (f''/f')' - (1/2)(f''/f')^2$ with univalence. In a recent paper [HH], the issues of continuous extension to $\overline{\mathbb{D}}$ and extremal behavior were analyzed under an assumption weaker than (1), namely that

$$\sup_{r,\zeta}(1-r^2)^2\sigma_f(r,\zeta) \leq 2\,, \tag{2}$$

where $0 \leq r < 1$, $|\zeta| = 1$, and

$$\sigma_f(r,\zeta) = \mathrm{Re}[\zeta^2 Sf(r\zeta)] - \frac{1}{2}[\mathrm{Im}\{\zeta\frac{f''}{f'}(r\zeta)\}]^2\,. \tag{3}$$

The class NH of functions locally univalent in $\mathbb{D}$ and satisfying (2) is invariant only under affine changes in the range and rotations in the disk. Simple examples show that they may not be univalent in $\mathbb{D}$, not even on subdisks of arbitrarily small, fixed radius. Despite the rather complicated

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expression for σ_f, this quantity appears in a natural way when applying Sturm comparison techniques to study problems of distortion related to Sf (see, e.g., [COP] and [GP]). As we will see, condition (2) has the consequence that the function

$$h_f(z) = \frac{1}{\sqrt{(1-|z|^2)|f'(z)|}}$$

is convex along every ray from the origin parametrized with the hyperbolic arclength parameter $s = \frac{1}{2}\log\frac{1+r}{1-r}$ (see [COP] for a similar characterization of the class N). In [HH, Proposition 5.3], the authors consider an extension of a well known result of Gehring and Pommerenke, and show that functions in NH with $f''(0) = 0$ are either unbounded or else bounded and having a logarithmic modulus of continuity in $\overline{\mathbb{D}}$. They leave open the question as to what the unbounded functions can be. Using the above mentioned property of the function h_f, we will show that such an f must be an affine transformation of a rotation of $L(z) = \frac{1}{2}\log\frac{1+z}{1-z}$. It is an unresolved problem in the same Proposition to determine whether the functions in the bounded case remain univalent in $\overline{\mathbb{D}}$.

We will also consider in this paper the subclasses NH_t, $0 \le t < 1$, given by

$$\sup_{r,\zeta}(1-r^2)^2\sigma_f(r,\zeta) \le 2t\,, \tag{4}$$

and will derive sharp distortion theorems both for NH and NH_t (see, [CO1]). In order to study John disks we will consider the asymptotic bound

$$\sup_{\zeta}\,[\limsup_{r\to 1}(1-r^2)^2\sigma_f(r,\zeta)] < 2\,,$$

together with a simple finiteness condition that, in some sense, replaces the normalization $f''(0) = 0$.

2. CONVEXITY AND DISTORTION

It is well known that a locally injective meromorphic function f can be written as the quotient U_1/U_2 of two linearly independent solutions of the linear equation $U'' + \frac{1}{2}(Sf)U = 0$, and that $U_0 = (f')^{-1/2}$ is always one particular solution. Somewhat tedious but straightforward calculations show that the restriction $u(r) = |U_0(r\zeta)|$ of $|U_0|$ to a ray $[0,\zeta)$ satisfies

$$u''(r) + \frac{1}{2}\sigma_f(r,\zeta)u(r) = 0\,. \tag{5}$$

In other words, when $f \in NH$ one will have information about the behavior of $|U|$ along rays for one solution of the above complex linear equation, and not necessarily for every such solution. The next comparison lemma will be crucial to our considerations. It can be found in [CO3], but we include the proof for convenience.

Lemma 1: *Let $p = p(r), q = q(r)$ be real valued functions on $[0,1)$ with $p(r) \le q(r)$. Let u, v be positive solutions of $u'' + pu = 0$ and $v'' + qv = 0$, respectively. Then*

$$w(s) = \frac{u}{v}(G(s))$$

is a convex function of s, where $G = F^{-1}$ and

$$F(r) = \int_0^r v^{-2}(x)dx\,. \tag{6}$$

Proof: Since $F' = v^{-2}$ we see that, at corresponding points, $G' = v^2$. Therefore $w' = vu' - uv'$ and $w'' = (vu'' - uv'')v^2 = (q - p)uv^3 \geq 0$.

Lemma 2: *If $f \in NH$ then h_f is convex along every ray $[0, \zeta)$ as a function of the hyperbolic arclength parameter.*

Proof: Let $f \in NH$. In virtue of (5) it follows for each ζ, the function $u(r) = |f'(r\zeta)|^{-1/2}$ satisfies an equation of the form $u''(r) + p(r)u(r) = 0$ with $p(r) \leq q(r) = 1/(1-r^2)^2$. A positive solution of $v'' + qv = 0$ is given by $v(r) = \sqrt{1 - r^2}$, which in (6) gives rise to the function $F(r) = L(r) = \frac{1}{2}\log\frac{1+r}{1-r}$. To finish the proof, simply observe that $w(s)$ is precisely the function $h_f(r(s)\zeta)$, where $r(s) = (e^{2s} - 1)/(e^{2s} + 1)$.

We derive from this the following theorem that completes the description of part (i) of Proposition 5.3 in [HH].

Theorem 1: *Let $f \in NH$ satisfy $f''(0) = 0$. If the image $\Omega = f(\mathbb{D})$ is unbounded then f is of the form*

$$f(z) = aL(e^{i\theta}z) + b\,.$$

Proof: The normalization $f''(0) = 0$ produces a critical point of the function h_f at $z = 0$. Because of the hyperbolic convexity along every ray from the origin, it follows that $h_f(0)$ is an absolute minimum of h_f. Simple considerations show that either

(a) there exist $s_0, \alpha > 0$ such that for all $\zeta, s \geq s_0$

$$\frac{d}{ds}h_f(r(s)\zeta) > \alpha\,,$$

or

(b) h_f is constant along some ray $[0, \zeta_0)$.

As it was done in [HH], the case (a) leads to bounds on $|f'(z)|$ that are integrable and which give the logarithmic modulus of continuity (see also [GP]). On the other hand, suppose (b) holds. After a rotation, we may assume that h_f is constant along the ray $[0, 1)$. This constant value is the minimum, hence the gradient of h_f vanishes there. In other words, for all $r \in [0, 1)$

$$\frac{\partial h_f}{\partial z}(r) = 0\,,$$

from which

$$\frac{f''}{f'}(r) = \frac{2r}{1 - r^2}\,.$$

From this, simple integration and analytic continuation show that $f = aL + b$, as claimed.

Remark 1: Let $f \in NH$ be univalent with $f''(0) = 0$, and let $\lambda(w)|dw|$ be the Poincaré metric of the image domain $\Omega = f(\mathbb{D})$. The argument in the proof of Theorem 1 shows that the critical point of $\lambda(w)$ at $w_0 = f(0)$ is unique unless f is of the form $aL + b$.

The next two theorems will be also consequences of the comparison principle in Lemma 1. We will consider the classes NH or NH_t and specific normalizations, to which end we introduce the functions

$$A_t(z) = \frac{1}{\sqrt{1-t}}\frac{(1+z)^{\sqrt{1-t}} - (1-z)^{\sqrt{1-t}}}{(1+z)^{\sqrt{1-t}} + (1-z)^{\sqrt{1-t}}}\,,$$

which satisfy $A_t(0) = 0, A_t'(0) = 1, A_t''(0) = 0$. They have $SA_t(z) = 2t/(1-z^2)^2$, and exhibit certain extremal behavior in the classes NH_t.

Theorem 2: *Let $f \in NH$ with $f(0) = 0$, $|f'(0)| = 1$ and $f''(0) = 0$. Then*

(*i*)
$$|f(r\zeta)| \leq L(r)\,,$$

(*ii*)
$$|f'(r\zeta)| \leq L'(r)\,,$$

(*iii*)
$$\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\} \leq \frac{L''}{L'}(r)\,.$$

If equality holds in any of these inequalities at some $r_0\zeta_0 \neq 0$ then f is of the form $f(z) = cL(e^{i\theta}z), |c| = 1$.

Theorem 3: *Let $f \in NH_t$ with $f(0) = 0$, $|f'(0)| = 1$ and $f''(0) = 0$. Then*

(*i*)
$$|f(r\zeta)| \leq A_t(r)\,,$$

(*ii*)
$$|f'(r\zeta)| \leq A_t'(r)\,,$$

(*iii*)
$$\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\} \leq \frac{A_t''}{A_t'}(r)\,.$$

If equality holds in any of these inequalities at some $r_0\zeta_0 \neq 0$ then f is of the form $f(z) = cA_t(e^{i\theta}z), |c| = 1$.

The proof of both theorems is essentially the same, and we will prove only Theorem 3.

Proof: Since $SA_t(z) = 2t/(1-z^2)^2$ it follows that $v(r) = (A_t'(r))^{-1/2}$ is a (positive) solution of $v'' + qv = 0$, where $q(r) = t/(1-r^2)^2$. Observe also that because of the normalizations of A_t, $v(0) = 1$ and $v'(r) = 0$. Let $f \in NH_t$ be normalized as in the theorem. It follows from (5) that for each ζ, the function $u(r) = |f'(r\zeta)|^{-1/2}$ satisfies an equation of the form $u'' + pu = 0$ with $p(r) \leq q(r)$. Also, $u(0) = 1$ and $u'(0) = 0$. We apply now Lemma 1 to conclude that, for each ζ,

$$w(s) = \sqrt{\frac{A_t'(r(s))}{|f'(r(s)\zeta)|}}$$

is a convex function of s, where $ds/dr = (A_t'(r))^{-2}$. Since $w(0) = 1$ and $w'(0) = 0$, we conclude as in the proof of Theorem 1, that the function w attains its minimum at 0. Hence $w(s) \geq 1$, which gives (*ii*). Direct integration gives now (*i*). To obtain (*iii*), simply observe that $w'(s) \geq 0$, thus

$$0 \leq 2\frac{w'}{w}(s) = \left[\frac{A_t''}{A_t'}(r) - \mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\}\right]\frac{dr}{ds}\,. \tag{7}$$

Finally, we discuss the cases of equality at some $r_0\zeta_0 \neq 0$. It is clear that such a case of equality in (*i*) must come from equality in (*ii*) for all $r\zeta_0$, $r \leq r_0$, and similarly, that equality in (*ii*) at some $r_0\zeta_0$ implies equality in (*iii*) along the segment $[0, r_0\zeta_0]$. Therefore it suffices to consider just the case of equality in (iii). After replacing $f(z)$ with $f(\zeta_0 z)$, we may assume that $\zeta_0 = 1$. We see from (7) that $w'(s_0) = 0$, and because $w(s)$ is convex and $w'(0) = 0$, we have that $w'(s) = 0$ for all $0 \le s \le s_0$. But then $w(s) = 1$ for all such s, and we conclude that the function

$$H_f(z) = \sqrt{\frac{A_t'(|z|)}{|f'(z)|}}$$

attains its minimum value in the disk on the segment $[0, r_0]$. Hence for $0 \le r \le r_0$

$$\frac{\partial H_f}{\partial z}(r) = 0\,,$$

which gives

$$\frac{f''}{f'}(r) = \frac{A_t''}{A_t'}(r)\,.$$

From here, analytic continuation, direct integration and the use of the normalization at the origin, give that $f = cA_t$ for some $|c| = 1$.

Remark 2: We mention that

$$A_t'(r) = \mathrm{O}\left(\frac{1}{(1-r^2)^{1-\sqrt{1-t}}}\right)\,,$$

and that

$$\frac{A_t''}{A_t'}(r) = \frac{2(r - \sqrt{1-t}A_t(r))}{1-r^2} \le \frac{2tr}{1-r^2}\,. \tag{8}$$

3. JOHN DISKS

In this section we will study the concept of a John disk in connection with the condition

$$\sup_{\zeta}\,[\limsup_{r\to 1}(1-r^2)^2\sigma_f(r,\zeta)] < 2\,. \tag{9}$$

Recall that a bounded, simply connected domain Ω is said to be John if there exists a constant M such that for every cross-cut C of Ω

$$\operatorname{diam} H \le M \operatorname{diam} C$$

holds for one of the components of $\Omega\backslash C$ (see, e.g. [P]). Geometrically, this condition prevents $\partial\Omega$ from having outward pointing cusps. See also Theorem 5.2 in [P] for a characterization of John domains in terms of the growth of the derivative of the conformal mapping onto Ω.

In [HH], the authors show that the image $\Omega = f(\mathbb{D})$ of a univalent mapping f is a John disk provided that

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\left\{z\frac{f''}{f'}(z)\right\} < 2\,, \tag{10}$$

extending a similar characterization of John domains within the Nehari class ([COP]). In particular, the authors derive from (10) bounds on $|f'|$ that are integrable, establishing in this way that the image Ω is bounded.

To begin with, we need following comparison lemma.

Lemma 3: *Let u, v be positive solutions of $u'' + pu = 0$ and $v'' + qv = 0$, respectively, and suppose that $p(r) \le q(r)$ for $r_0 \le r < 1$. If*

$$\lim_{r\to 1}(1-r)u^{-2}(r) = 0 \tag{11}$$

then

$$\limsup_{r\to 1}\left[-(1-r)\frac{u'}{u}(r)\right] \le \limsup_{r\to 1}\left[-(1-r)\frac{v'}{v}(r)\right]. \tag{12}$$

Remark 3: The conclusion of the lemma is not true without the condition (11). For example, let $u(r) = 1-r$, for which $p(r) = 0$, and let $v(r) = \sqrt{1-r^2}$, for which $q(r) = 1/(1-r^2)^2$. Then

$$\lim_{r\to 1}\left[-(1-r)\frac{v}{v}(r)\right] = \frac{1}{2} < 1 = \lim_{r\to 1}\left[-(1-r)\frac{u}{u}(r)\right].$$

Proof: The most general solution of the equation $w'' + pw = 0$ is given by $w = (ah+b)u$, where a, b are constants and $h(r) = \int_0^r u^{-2}(x)dx$. Thus

$$\frac{w'}{w} = \frac{u'}{u} + \frac{ah'}{ah+b}.$$

We claim that for appropriate choices of a, b

$$\frac{w'}{w}(r_0) \ge \frac{v'}{v}(r_0).$$

Indeed, this is equivalent to

$$\frac{ah'(r_0)}{ah(r_0)+b} = \frac{v'}{v}(r_0) - \frac{u'}{u}(r_0),$$

which can be accomplished by choosing, for example, $a = 1$ and b such that $h(r_0) + b$ is positive but sufficiently small. Note that this will guarantee also that $h(r) + b$ remains positive for $r \ge r_0$ because this function is increasing. For the constants a, b chosen in this way, the Sturm comparison theorem implies that $(w'/w)(r) \ge (v'/v)(r)$ for $r \in [r_0, 1)$, hence

$$-\frac{u'}{u}(r) - \frac{h'(r)}{h(r)+b} \le -\frac{v'}{v}(r). \tag{13}$$

The lemma now follows from the fact that $\lim_{r\to 1}(1-r)h'(r) = \lim_{r\to 1}(1-r)u^{-2}(r) = 0$.

In the applications we will take $u(r) = |f'(r\zeta)|^{-1/2}$, so that for bounded univalent f, (11) will be a consequence of the classical estimates for the hyperbolic metric. For univalent functions either in NH or satisfying (9), the assumption that f be bounded can be relaxed, as the next theorems show.

Theorem 4: *Let $f \in NH$ be univalent. If $\Omega = f(\mathbb{D})$ has finite area then f is bounded and admits a continuous extension to $\overline{\mathbb{D}}$ with a logarithmic modulus of continuity. Furthermore, for all $|\zeta| = 1$,*

$$\int_0^1 |f'(r\zeta)|dr < \infty. \tag{14}$$

Theorem 5: *Let $f : \mathbb{D} \to \mathbb{C}$ be univalent and satisfy (9). If $\Omega = f(\mathbb{D})$ has finite area then f is bounded and admits a $\sqrt{1-t}$-Hölder continuous extension to $\overline{\mathbb{D}}$. Furthermore, for all $|\zeta| = 1$,*

$$\int_0^1 |f'(r\zeta)| dr < \infty \,. \tag{15}$$

Again, the proof of both theorems is essentially the same, and we prove only Theorem 5.

Proof: If f satisfies (9) then there exists a fixed $t < 1$ so that for each ζ there exists an $r_0 = r_0(\zeta) < 1$ such that for $r \geq r_0(\zeta)$

$$\sigma_f(r, \zeta) < \frac{t}{(1-r^2)^2} \,. \tag{16}$$

It is easy to see that $r_0(\zeta)$ can be chosen so that $r_0(\zeta)$ remains locally uniformly bounded away from 1. By compactness, we can find $r_0 < 1$ such that (16) holds for all $r \geq r_0$ and all ζ. Therefore, the function

$$h_f(r\zeta) = \frac{1}{\sqrt{(1-r^2)|f'(r\zeta)|}}$$

is convex with respect to the hyperbolic parametrization $s = L(r)$ of every radial segment $[r_0\zeta, \zeta)$. We claim that h_f cannot be decreasing along an entire radial segment. If so, then $h_f(r\zeta_0) \leq c$, hence

$$|f'(r\zeta_0)| \geq \frac{c^{-2}}{1-r^2} \,. \tag{17}$$

After a rotation, we may assume that $\zeta_0 = 1$, and let us consider the region $R = \{z \in \mathbb{D} : \mathrm{d}_h(z, [r_0, 1)) \leq \epsilon\}$. We claim that $f(R)$ has infinite area, which will give a contradiction. To show this, observe that the hyperbolic distance between the points r and $r + iy$ is given by $L(q) = \frac{1}{2} \log \frac{1+|q|}{1-|q|}$, where $q = |y|/((1-r^2)^2 + r^2y^2)^{1/2}$. Let $\epsilon = L(\delta)$. It follows that, for $r \geq r_0$, the point $r + iy$ will belong to R provided $|y| \leq y(r)$, where

$$\delta = \frac{y(r)}{\sqrt{(1-r^2)^2 + r^2y^2(r)}} \leq \frac{y(r)}{1-r^2} \,,$$

hence $y(r) \geq \delta(1-r^2)$. Since for univalent mappings $|f'(w)|/|f'(z)| \leq e^{6\mathrm{d}_h(z,w)}$, it follows now from (17) that

$$\int\int_R |f'|^2 dA = \infty \,.$$

We conclude from this that for each ζ there exists $r_1(\zeta)$ such that $(dh_f/ds)(r_1\zeta) > 0$. Because of continuity, $r_1(\zeta)$ can be chosen locally uniformly bounded away from 1, so that by compactness, there exists r_1 independent of ζ such that

$$(dh_f/ds)(r_1\zeta) \geq \alpha > 0 \,, \tag{18}$$

for all ζ. The bound on σ_f implies a strong degree of convexity. Indeed, in applying Lemma 1 to $u(r) = |f'(r\zeta)|^{-1/2}$ and $v(r) = \sqrt{1-r^2}$, and corresponding functions $p(r) = \sigma_f(r, \zeta)$, $q(r) = 1/(1-r^2)^2$, we have that the function $w(s) = (u/v)(r(s)) = h_f(r(s)\zeta)$ satisfies

$$w'' = (q-p)uv^3 = (q-p)v^4 w \geq (1-t)w \,. \tag{19}$$

In light of (18), it follows that for $s \geq s_1 = L(r_1)$,

$$h_f(r(s)\zeta) \geq a e^{s\sqrt{1-t}},$$

where the constant a is positive and only depends on α, s_1 and $\min_\zeta h_f(r_1\zeta)$. This gives that for $|z| \geq r_1$

$$\frac{1}{(1-|z|^2)|f'(z)|} \geq a^2 \left(\frac{1+|z|}{1-|z|}\right)^{\sqrt{1-t}},$$

which implies

$$|f'(z)| \leq \frac{a^{-2}}{(1+|z|)^{\sqrt{1-t}}} \frac{1}{(1-|z|)^{1-\sqrt{1-t}}} \leq \frac{a^{-2}}{2(1-|z|)^{1-\sqrt{1-t}}}. \tag{20}$$

From this, the well known argument based on integrating $|f'|$ along hyperbolic segments, shows that f admits a Hölder continuous extension to $\mathbb{D}$ with exponent $\sqrt{1-t}$ (see, e.g., [GP]). In addition, it follows immediately from (20) that (15) will hold for all ζ. This finishes the proof of Theorem 6.

With this, we can now establish the following result.

Theorem 6: *Let $f : \mathbb{D} \to \mathbb{C}$ be univalent and satisfy (9). If $\Omega = f(\mathbb{D})$ has finite area then*

$$\limsup_{|z|\to 1} (1-|z|^2)\mathrm{Re}\left\{z\frac{f''}{f'}(z)\right\} < 2, \tag{21}$$

and Ω is a bounded John disk.

Proof: We already know from Theorem 6 that Ω is bounded. Therefore, for all ζ, $\lim_{r\to 1}(1-r^2)|f'(r\zeta)| = 0$. As we showed in Theorem 6, there exist r_0 such that for $r \geq r_0$,

$$\sigma_f(r,\zeta) \leq \frac{t}{(1-r^2)^2}.$$

We apply now Lemma 2 with $u(r) = |f'(r\zeta)|^{-1/2}$, $v(r) = (A_t'(r))^{-1/2}$, and functions $p(r) = \sigma_f(r,\zeta)$, $q(r) = t/(1-r^2)^2$, to conclude that

$$\limsup_{r\to 1}(1-r^2)\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\} \leq \limsup_{r\to 1}(1-r^2)\frac{A_t''}{A_t'}(r) \leq 2t. \tag{22}$$

In order to prove the theorem, we need to show that the bound (13) in Lemma 3 leading to (22) can be made uniform in ζ. The constant b in (13) must be chosen so that for all ζ

$$\frac{|f'(r_0\zeta)|}{\int_0^{r_0}|f'(x\zeta)|dx + b} \geq -\frac{A_t''}{2A_t'}(r_0) + \mathrm{Re}\left\{\frac{f''}{2f'}(r_0\zeta)\right\},$$

which can be accomplished because the right hand side is bounded in ζ and because both $|f'(r_0\zeta)|$ and $\int_0^{r_0}|f'(x\zeta)|dx$ are positive and bounded in ζ. In other words, for a fixed b, we see from (13) that for all $r \geq r_0$ and all ζ,

$$\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\} \leq \frac{A_t''}{A_t'}(r) + \frac{|f'(r\zeta)|}{\int_0^r|f'(x\zeta)|dx + b} \leq \frac{2t}{1-r^2} + \frac{|f'(r\zeta)|}{\int_0^r|f'(x\zeta)|dx + b}.$$

The inequality (21) is now a consequence of (20). The fact that Ω is a John disk follows now from Theorem 3.7 in [HH].

4. AN EXAMPLE

We show by means of an example that function in NH are, in general, not univalent. Let

$$f(z) = e^{cz} \, .$$

Then $f''/f' = c$ and $Sf = -c^2/2$, so that

$$\sigma_f(r\zeta) = -\frac{1}{2}\mathrm{Re}\{c^2\zeta^2\} - \frac{1}{2}[\mathrm{Im}\{c\zeta\}]^2 = -\frac{1}{2}\mathrm{Re}\{c^2\zeta^2\} + \frac{1}{2}\mathrm{Re}\{c^2\zeta^2\} - \frac{1}{2}[\mathrm{Re}\{c\zeta\}]^2 \leq 0 \, .$$

Hence $f \in NH$ for all values of c, but it will fail to be univalent in $\mathbb{D}$ if $|c| > \pi$. In fact, $f(-\frac{\pi}{c}) = f(\frac{\pi}{c})$, thus the radius of univalence tends to 0 as $|c| \to \infty$.

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